Improved analysis of the renormalization scheme ambiguities in the QCD corrections to the semileptonic decay of the tau lepton[*]

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Abstract

The perturbative QCD corrections to the semileptonic decay width of the tau lepton are evaluated in the next-next-to-leading order from the contour integral representation in various renormalization schemes, using numerical solution of the renormalization group equation for complex energies. A quantitative estimate of the ambiguities resulting from the freedom of choice of the renormalization scheme is obtained by taking into account predictions in all schemes that do not involve large cancellations in the expression for the scheme invariant combination of the expansion coefficients. The problem of an optimal choice of the renormalization scheme for the improved perturbative expression is discussed. A fit of $\Lambda_{\overline{MS}}$ is made using the available experimental data.

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As is well known, predictions of a quantum field theory obtained with finite order perturbative expressions depend on the choice of the renormalization scheme (RS). This effect is usually regarded as a complication in the comparison of the theory with the experimental data. However, the effect of renormalization scheme dependence may be also used to improve our understanding of the perturbative result. By estimating the change in the predictions over a set of *a priori* acceptable schemes we may test reliability of the perturbative expression. This point of view has been recently discussed in [1, 2], where a concrete method has been proposed for a systematic evaluation of the RS ambiguities in the QCD predictions in the next-next-to-leading order (NNLO). This method is based on the observation, that one should take into account predictions in all schemes that do not involve large cancellations in the expression for scheme independent combination of the expansion coefficients. Application of this method was illustrated in [2] using as an example the QCD corrections to the Bjorken sum rule for the polarized structure functions. Also the problem of the RS ambiguities in the QCD corrections to the ratio [3]-[10]:

$$R_\tau = \frac{\Gamma(\tau \to \nu_\tau + hadrons)}{\Gamma(\tau \to \nu_\tau e^- \overline{\nu}_e)}, \tag{1}$$

has been considered in this framework [1]. In this case it was found, that even small changes in the RS - among acceptable schemes - resulted in a large variation in the predictions. The strong RS dependence seemed to undermine the phenomenological importance of the QCD corrections to R_τ.

However, the QCD correction to R_τ comes originally in the form of a contour integral in the complex energy plane [3]-[7]. The commonly used perturbative formula appears as a result of approximate evaluation of the contour integral. As was pointed out in [11, 12], the evaluation of the contour integral may be improved. In [12] it was shown that using the improved evaluation of the contour integral one obtains smaller scale dependence in the $\overline{MS}$ scheme. It is then an interesting problem, how the improved accuracy of the predictions affects the estimate of the strength of the RS dependence evaluated according to the method developed in [1, 2].

In this note we evaluate the QCD corrections to R_τ in various renormalization schemes by solving numerically the renormalization group equation for complex energies, and by computing numerically the relevant contour integral. Using the method developed in [1, 2] we obtain a quantitative estimate of the strength of the RS dependence in this case [13]. We discuss the problem of an optimal choice of the renormalization scheme in the improved expression. We also make a fit of $\Lambda_{\overline{MS}}$ taking into account recently obtained experimental results for R_τ.

Let us begin with a brief summary of the ideas presented in [1, 2]. Let us consider a NNLO expression for a physical quantity δ, properly normalized, depending on a single energy variable P:

$$\delta(P^2) = a(P^2)[1 + r_1 a(P^2) + r_2 a^2(P^2)], \tag{2}$$

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where $a(\mu^2) = g^2(\mu^2)/(4\pi^2)$ denotes the running coupling constant that satisfies the NNLO renormalization group equation:

$$\mu\frac{da}{d\mu} = -b\,a^2\,(1 + c_1 a + c_2 a^2\,), \tag{3}$$

(The effects of the quark masses are neglected.) The value of the predictions obtained for δ with the NNLO expressions (2) and (3) depends on the choice of RS. Presently in the phenomenological applications the modified minimal subtraction scheme ($\overline{MS}$) [14] is used. However, other choices of the renormalization scheme are possible, all related to the $\overline{MS}$ scheme by a finite renormalization. In the NNLO, in the class of the mass and gauge independent schemes, the freedom of choice of the scheme may be characterized by two continuous parameters. (The relevant formulas describing the RS dependence of the coefficients r_1, r_2 and c_2 have been collected in [1].) The consistency of the perturbation theory guarantees, of course, that differences in the predictions obtained in various schemes are formally of higher order in the coupling constant, but numerical magnitude of these differences is significant for phenomenological analysis. This fact stimulated the interest in prescriptions for making a proper choice of the RS [15]-[24]. The most attractive of these propositions seems to be the choice based on the so called Principle of Minimal Sensitivity (PMS) [17].

However, whatever scheme we decide to choose as an optimal one, there is a continuum of equally reasonable schemes close to the one preferred by us. Predictions in such schemes also should be somehow taken into account in the phenomenological analysis. A natural way to do this is to supplement the prediction in a preferred scheme with an estimate of the variation of the predictions over the whole set of *a priori* acceptable schemes. A concrete realization of this idea was presented in [2], based on the existence of an RS independent combination of the expansion coefficients [17, 21, 22, 23]:

$$\rho_2 = c_2 + r_2 - c_1 r_1 - r_1^2. \tag{4}$$

which provides a natural RS independent characterization of the magnitude of the NNLO corrections for the considered physical quantity. In [2] it was proposed to calculate variation of the predictions for δ over the set of schemes for which the expansion coefficients satisfy the condition:

$$\sigma_2(r_1, r_2, c_2) \leq l\,|\rho_2|, \tag{5}$$

where

$$\sigma_2(r_1, r_2, c_2) = |c_2| + |r_2| + c_1|r_1| + r_1^2. \tag{6}$$

A motivation for the condition (5) is that it eliminates schemes in which the expressions (2) and (3) involve unnaturally large expansion coefficients, implying large cancellations in the expression for the RS invariant ρ_2. The constant l in the condition (5) controls the degree of cancellations that we want to allow in the expression

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for ρ_2. As was pointed out in [2], taking $l = 2$ in (5) one has the PMS scheme right at the boundary of the allowed region in the space of parameters characterizing the scheme.

It is expected that the estimate of the strength of the RS dependence obtained according to this prescription would be useful for a quantitative comparison of reliability of perturbative predictions for different physical quantities, evaluated at different energies. It should be also very useful in determining the regions of applicability of the perturbative expansion. Indeed, any large variation of the predictions over a set of schemes satisfying constraint (5) with $l = 2$ would be an unambiguous sign of a limited applicability of the NNLO expression.

Let us now consider the QCD corrections to the semileptonic decay of the tau lepton. These corrections, denoted further by δ_τ, contribute to the ratio R_τ in the following way:

$$R_\tau = 3(|V_{ud}|^2 + |V_{us}|^2)S_{EW}(1 + \tilde{\delta}_{EW} + \delta_\tau) \tag{7}$$

where V_{ij} are the Cabbibo-Kobayashi-Maskawa matrix elements and $S_{EW} = 1.0194$, $\tilde{\delta}_{EW} = 0.001$ are factors arising from the electroweak corrections [25, 26]. The perturbative correction to δ_τ is known up to NNLO [5]-[8] in the approximation of massless u, d and s quarks, and it has in the conventional approach the form (2) with the characteristic energy scale $P = m_\tau = 1.7771$ GeV.

The interest in the QCD corrections to R_τ [5]-[12], [27]-[41] was originally stimulated by the fact, that the perturbative expression for δ_τ appeared to be quite sensitive to the QCD parameter Λ. Thus in principle a very good test of perturbative QCD predictions could be obtained [5, 6], since the effects of nonzero masses of light quarks, and the nonperturbative effects, were estimated to be small despite the rather low energy scale. Unfortunately, in [1] large differences in the perturbative predictions for δ_τ have between obtained in *a priori* acceptable schemes. (The perturbative QCD corrections to R_τ in some optimal schemes were considered previously in [27, 28, 31, 32].) It seemed therefore that the high sensitivity of δ_τ to the QCD parameter Λ is practically eliminated by the ambiguities introduced by the strong RS dependence [42].

However, the commonly used perturbation expansion for δ_τ involves not only the obvious approximation resulting from the truncation of the perturbation series, but also an approximation of another kind, which appears in the process of evaluation of δ_τ from the expression for the two-point quark current correlators. Indeed, in order to evaluate the QCD corrections to R_τ one starts from the formula [3, 4, 5, 7]:

$$R_\tau = 12\pi \int_0^{m_\tau^2} \frac{ds}{m_\tau^2} \left(1 - \frac{s}{m_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{m_\tau^2}\right) Im\Pi^{(1)}(s + i\epsilon) + ...\right], \tag{8}$$

where $\Pi^{(1)}$ denotes the sum of the transverse parts of the $\Delta S = 0, 1$ vector and axial quark current correlators:

$$\Pi^{(1)}(s) = |V_{ud}|^2[\Pi_{ud,V}^{(1)}(s) + \Pi_{ud,A}^{(1)}(s)] + |V_{us}|^2[\Pi_{us,V}^{(1)}(s) + \Pi_{us,A}^{(1)}(s)], \tag{9}$$

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$$\Pi^{\mu\nu}_{kl,V/A}(q) = (-g^{\mu\nu}q^2 + q^\mu q^\nu)\Pi^{(1)}_{kl,V/A}(q^2) + ..., \tag{10}$$

$$\Pi^{\mu\nu}_{kl,V/A}(q) = i \int d^4x e^{iqx} < 0|T(J^\mu_{kl,V/A}(x)J^\nu_{kl,V/A}(0)^\dagger)|0 > . \tag{11}$$

(If the quark mass effects are neglected, the longitudinal part of $\Pi^{\mu\nu}$ does not contribute. Also, the electroweak contributions have been neglected.) Since the QCD predictions for the quark current correlators are not very well known for real positive s, one uses the analyticity properties of Π to convert the expression (8) into the contour integral in the complex energy plane:

$$R_\tau = \frac{6\pi}{i} \int_C \frac{ds}{m_\tau^2}\left(1 - \frac{s}{m_\tau^2}\right)^2 \left[\left(1 + 2\frac{s}{m_\tau^2}\right)\Pi^{(1)}(s) + ...\right], \tag{12}$$

where C is a contour running clockwise from $s = m_\tau^2 - i\epsilon$ to $s = m_\tau^2 + i\epsilon$ and avoiding the region of small $|s|$, i. e. on C we have $|s| = m_\tau^2$. Integrating by parts along the contour, one obtains:

$$\delta_\tau = \frac{1}{2\pi} \int_{-\pi}^{\pi} d\theta \left(1 + 2e^{i\theta} - 2e^{3i\theta} - e^{4i\theta}\right)\left[\delta_\Pi(-s)|_{s=-m_\tau^2 e^{i\theta}}\right], \tag{13}$$

where $\delta_\Pi(-s)$ is defined via the relation:

$$(-12\pi^2)s\frac{d}{ds}\Pi^{(1)}_V(s) = 3(|V_{ud}|^2 + |V_{us}|^2)[1 + \delta_\Pi(-s)]. \tag{14}$$

The importance of this expression lies in the fact, that δ_τ has been expressed via the quantity $\delta_\Pi(-s)$, which is formally RS independent, and which is directly calculable in perturbative QCD for real negative s. $\delta_\Pi(-s)$ has a perturbation expansion of the form (2), with the expansion coefficients in the $\overline{MS}$ scheme, with $\mu^2 = -s$, given by [8]:

$$r_1^{\overline{MS}} = 1.63982, \quad r_2^{\overline{MS}} = 6.37101. \tag{15}$$

The commonly used expansion of δ_τ in terms of $a(m_\tau^2)$ is obtained from the expression (13) by expanding $a(-s)$ under the integral in terms of $a(m_\tau^2)$:

$$a(-s) = a(m_\tau^2)\left[1 - \frac{b}{2}\ln\left(-s/m_\tau^2\right)a(m_\tau^2)+\right.$$

$$\left. + \left(\left(\frac{b}{2}\ln\left(-s/m_\tau^2\right)\right)^2 - \frac{b}{2}\ln\left(-s/m_\tau^2\right)\right)a^2(m_\tau^2)\right], \tag{16}$$

and performing the contour integral over the appearing powers of $\ln(-s/m_\tau^2)$ explicitly. In this way one obtains the NNLO expression for δ_τ which has the form (2) with the expansion coefficients $\tilde{r}_i$:

$$\tilde{r}_1 = r_1 + \frac{19}{24}b, \tag{17}$$

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$$\tilde{r}_2 = r_2 + \frac{19}{12}\, b\, r_1 + \frac{19}{24}\, b\, c_1 + \frac{265 - 24\pi^2}{288}\, b^2. \tag{18}$$

In the $\overline{MS}$ scheme we have [8]:

$$\tilde{r}_1^{\overline{MS}} = 5.2023, \quad \tilde{r}_2^{\overline{MS}} = 26.366. \tag{19}$$

We see that there is a significant difference between the expansion coefficients r_i and $\tilde{r}_i$.

It should be noted however, that using the expansion of $a(-s)$ in terms of $a(m_\tau^2)$ in the NNLO expression for δ_Π, we make a rather crude approximation. For example, we effectively assume, that:

$$a^3(-s) \approx a^3(m_\tau^2). \tag{20}$$

As was pointed out in [11, 12] one may obtain a more accurate description of the QCD effects in the tau lepton decay by evaluating the s-dependence of the running coupling constant on the contour C in a more precise way.

It is then an interesting question, whether the improved evaluation of the QCD corrections may change the rather upsetting result found in [1].

In this note we analyze the predictions for δ_τ obtained from the contour integral representation (13) using a numerical solution of the renormalization group equation in the complex energy plane. To obtain the running coupling constant along the contour C we solve numerically the implicit equation:

$$\frac{b}{2} \ln\left(\frac{m_\tau^2}{\Lambda_{\overline{MS}}^2}\right) + i\frac{b\theta}{2} = r_1^{\overline{MS}} - r_1 + \Phi(a, c_2), \tag{21}$$

where

$$\Phi(a, c_2) = c_1 \ln\left(\frac{b}{2c_1}\right) + \frac{1}{a} + c_1 \ln(c_1 a) + O(a). \tag{22}$$

The explicit form of $\Phi(a, c_2)$ is given for example in [43]. (The renormalization group coefficients for $n_f = 3$ are $b = 4.5$, $c_1 = 16/9$ and $c_2^{\overline{MS}} = 3863/864 \approx 4.471$.) This equation is obtained by integrating the renormalization group equation (3) with an appropriate boundary condition [14], and then analytically continuing to general complex s. (The presence of $r_1^{\overline{MS}}$ and $\Lambda_{\overline{MS}}$ in this general expression results from taking explicitly into account the one-loop relation between Λ parameters in different schemes [15], which is valid to all orders of perturbation expansion. All values of $\Lambda_{\overline{MS}}$ mentioned in this paper correspond to three "active" quark flavors.) To obtain the prediction for δ_τ we take under the contour integral the quantity δ_Π in the NNLO approximation in the form (2) and we perform the contour integral numerically. To analyze the RS dependence of the thus obtained expression for δ_τ we parametrize the freedom of choice of the RS by the coefficient r_1 in δ_Π and the coefficient c_2, and we calculate variation in the predictions when these parameters are changed in a region determined by the condition (5). The region of allowed

values for r_1 and c_2 for $\rho_2 > c_1^2/4$, which is the case needed here, was described analytically in [2].

In our analysis a fundamental role is played by the NNLO RS invariant, which for δ_Π has the value $\rho_2 = 5.238$. This value should be compared with $\tilde{\rho}_2 = -5.476$, which is obtained for the conventional expansion of δ_τ in terms of $a(m_\tau^2)$. The fact that $\tilde{\rho}_2$ has similar magnitude, but the opposite sign, compared to ρ_2 indicates that the approximations used to derive the conventional expansion for δ_τ may greatly distort the pattern of the RS dependence of the predictions.

The results of our calculations are presented in four figures. In Fig.1 we compare perturbative QCD predictions for δ_τ as a function of $m_\tau/\Lambda_{\overline{MS}}$, obtained in the next-to-leading order (NLO) and in NNLO using the conventional expansion, with the predictions obtained by evaluating numerically the contour integral. We clearly see that the differences between NLO and NNLO are much smaller for the predictions obtained from numerical calculation of the contour integral than for the predictions obtained with the conventional expansion in terms of $a(m_\tau^2)$. This confirms the expectation that the procedure for evaluating the δ_τ adopted in this note gives indeed improvement over the conventional expansion.

In Fig.2 we show a contour plot for δ_τ as a function of r_1 and c_2, obtained for $\Lambda_{\overline{MS}} = 310$ MeV, together with the region of scheme parameters satisfying the condition (5) with $l = 2$ — smaller region — and $l = 3$. (Note that — for purely technical reasons — the actual variable on the vertical axis is $c_2 - c_1 r_1$ instead of c_2.) We see that the pattern of RS dependence in this case has more complicated structure than that obtained for conventional NNLO approximants — this is evident if one compares Fig.2 with the corresponding figure in [2], representing the RS dependence of the QCD corrections to the Bjorken sum rule. In particular, we find here *three* critical points which are relatively close to the allowed region, instead of one encountered in the conventional approximant. The coordinates of the critical point closest to the origin, i.e. the saddle point at the boundary of the $l = 2$ allowed region, are very close to the PMS parameters for the quantity δ_Π evaluated for real negative s ("euclidean" PMS). (Let us note that because of a nonpolynomial character of the improved expression for δ_τ the value of the PMS parameters cannot be obtained from the set of algebraic equations given in [17], but it has to be determined from direct numerical analysis.) This confirms the fundamental importance of δ_Π for δ_τ. It also shows that approximations used to derive the conventional NNLO expression greatly distort the pattern of the RS dependence — the conventional NNLO approximant for δ_τ would have a critical point for completely different values of the scheme parameters. To estimate the strength of the RS dependence we compare predictions for δ_τ obtained for the scheme parameters lying in the denoted allowed region. Because of the location of the critical points in this case the maximal and minimal values for δ_τ are attained at the boundaries of the allowed regions corresponding to $l = 2$ and $l = 3$. It is interesting to note that the $\overline{MS}$ scheme with $c_2 - c_1 r_1 \approx 1.56$ lies slightly outside the $l = 3$ allowed region.

In Fig.3 we show, how the pattern of the RS dependence of the improved ex-

pression for δ_τ depends on the value of $\ln(m_\tau/\Lambda_{\overline{MS}})$. We see that for small values of $\Lambda_{\overline{MS}}$ we have only one critical point in the $l = 2$ allowed region, and that it lies very close to the PMS parameters for δ_Π evaluated for spacelike momenta. With increasing $\Lambda_{\overline{MS}}$ the structure of the critical points close to the $l = 2$ allowed region becomes more complicated. For some values of $\Lambda_{\overline{MS}}$ three critical points appear. For large values of $\Lambda_{\overline{MS}}$ the critical point corresponding to the "euclidean" PMS evolves into a very flat plateau, and the secondary critical point, with negative r_1, moves into the $l = 2$ allowed region.

In Fig.4, which is the main result of this paper, we show how the optimal perturbative predictions for δ_τ, and the estimates of the RS dependence, behave as a function of $m_\tau/\Lambda_{\overline{MS}}$. The results of our calculations are compared with constraints from the experimental data. As our preferred predition, indicated in Fig.4 by a thick solid line, we take the values obtained in the scheme with $r_1 = 0$, $c_2 = 1.5\rho_2$. These parameters describe, for small and intermediate values of $\Lambda_{\overline{MS}}$, the approximate location of the leading critical point in the (r_1, c_2) plane, and for larger values of $\Lambda_{\overline{MS}}$ they fall in the region of very small variation of the predictions. From Fig.3 it is clear, that for all interesting values of $\Lambda_{\overline{MS}}$ the predictions for δ_τ in this scheme practically coincide with the values at the relevant critical points. The dashed and dash-dotted lines represent the variation of the predictions over the $l = 2$ and $l = 3$ allowed regions, respectively. In order to make our results useful for other authors we give in Table 1 the numerical values of the preferred predictions for δ_τ, together with an estimate of the RS dependence, for several values of $\ln(m_\tau/\Lambda_{\overline{MS}})$. For completeness we include also the NLO predictions, evaluated using the contour integral and optimized according the the PMS prescription — they are well approximated by taking the scheme with $r_1 = -0.76$.

We see that the variation of the predictions over the $l = 2$ allowed region has a rather smooth dependence on $\Lambda_{\overline{MS}}$, being almost constant for the considered range of $\Lambda_{\overline{MS}}$. The variation over the $l = 3$ region of parameters is reasonably close to the $l = 2$ variation, except for the maximal value, which grows rapidly for larger $\Lambda_{\overline{MS}}$. This should be compared with the large differences found in [1] for the conventional NNLO approximant for δ_τ. We see that using an improved evaluation of the predictions for δ_τ we qualitatively improve stability of the predictions with respect to change of the RS, and we greatly reduce the ambiguities resulting from the freedom of choice of the renormalization scheme.

Let us now compare our results with those obtained in [12]. The analysis of [12] concentrates on variation of the renormalization scale μ in the $\overline{MS}$ scheme, which in our approach corresponds to variation of r_1 for fixed $c_2 = c_2^{\overline{MS}}$. In [12] only a brief comment is made on the variation of predictions when c_2 is changed from 0 to $2c_2^{\overline{MS}}$ with fixed μ. It must be stressed, that there is no theoretical or phenomenological reason justifying *a priori* such a choice of the set of the scheme parameters. Also, there is no immediate relation between the magnitude of variation found according to [12] and the estimate of the RS dependence obtained according to our method. In particular, small RS dependence obtained by prescription of

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[12] does not necessarily imply a small variation in our approach. It should be emphasized, that if we agree that there is a "democracy" of renormalization schemes, then the set of considered schemes must be larger than that discussed in [12], and any reasonable constraint on the freedom of choice must be related to the RS invariant ρ_2. Most importantly, a consideration of an essentially two-dimensional set of scheme parameters is unavoidable if one intends to find the optimal NNLO predictions according to the Principle of Minimal Sensitivity, which requires for the optimal scheme the vanishing of the derivatives with respect to *both* scheme parameters.

The results presented in Fig.4 may be used to obtain an improved constraint on $\Lambda_{\overline{MS}}^{(3)}$ from the world average of the available experimental data on R_τ. As is well known, the experimental value of R_τ may be obtained in three independent ways: from the electronic and muonic branching ratios in the tau lepton decay, and from the total width of the tau lepton. (Note that Particle Data Group gives *two* sets of values for the leptonic branching fractions — the number shown below corresponds to the set called "our average." See Appendix for a detailed discussion of the subtleties involved in obtaining experimental value for R_τ.) Taking a weighted average of these three determinations we find:

$$(R_\tau)_{exp}^{avg} = 3.591 \pm 0.036. \tag{23}$$

This value may then be converted into experimental constraint on δ_τ via Eq.(7). However, in order to have a phenomenologically meaningful constraint on δ_τ^{pert}, we must take into account also the effect of the light quark masses and the nonperturbative effects. According to [10, 37] these may be estimated to give roughly an overall $-(1.5 \pm 0.4)\%$ correction to the value of R_τ. Therefore, after taking into account that the factor involving the CKM matrix elements is practically equal to unity, our final formula relating R_τ and δ_τ^{pert} takes the form:

$$R_\tau = 3 \times 1.0194 \times (1.001 + \delta_\tau^{pert}) \times (0.985 \pm 0.004). \tag{24}$$

Using (23) we obtain:

$$(\delta_\tau^{pert})_{exp} = 0.191 \pm 0.012(\text{exp}) \pm 0.005(\text{npt}), \tag{25}$$

where the first error reflects the effect of experimental uncertainties and the second one the effect of uncertainties in nonperturbative corrections.

Fitting $\Lambda_{\overline{MS}}^{(3)}$ to this experimental value we obtain:

$$\Lambda_{\overline{MS}}^{(3)} = 376(\text{opt})_{-14}^{+15}(\text{th,l=2}) \pm 29(\text{exp}) \pm 12(\text{npt})\,\text{MeV}. \tag{26}$$

For the $l = 3$ region of parameters the variation in $\Lambda_{\overline{MS}}^{(3)}$ is $_{-21}^{+26}$ MeV.

As was explained in [1, 2], a natural way to parametrize the QCD predictions in our analysis is via the $\Lambda_{\overline{MS}}$. However, for comparison with results given in other papers on experimental tests of QCD, we give also the corresponding values

of $\alpha_s(m_\tau^2)$ in the $\overline{MS}$ scheme, obtained with the three-loop renormalization group equation:

$$\alpha_s^{\overline{MS}}(m_\tau^2) = 0.332^{+0.008}_{-0.007}(\text{th,l=2}) \pm 0.015(\text{exp}) \pm 0.006(\text{npt}). \tag{27}$$

For the $l = 3$ region of parameters we obtain for $\alpha_s^{\overline{MS}}(m_\tau^2)$ the variation of $^{+0.014}_{-0.010}$.

Using the formulas given in [45] to match $\Lambda_{\overline{MS}}$ for different number of flavors we obtain the corresponding result for $\alpha_s(m_Z^2)$:

$$\alpha_s^{\overline{MS}}(m_Z^2) = 0.1190^{+0.0009}_{-0.0008}(\text{th,l=2}) \pm 0.0017(\text{exp}) \pm 0.0007(\text{npt}). \tag{28}$$

For the $l = 3$ region of parameters we obtain for $\alpha_s^{\overline{MS}}(m_Z^2)$ the variation of $^{+0.0016}_{-0.0013}$.

We see that the RS dependence ambiguities in the determination of $\alpha_s^{\overline{MS}}(m_\tau^2)$ or $\alpha_s^{\overline{MS}}(m_Z^2)$ from R_τ are not very big — this is the result of stabilization of the predictions when improved evaluation procedure is used. Note however that the RS dependence ambiguities are still comparable in magnitude to the uncertainties related to the present experimental accuracy of R_τ.

Let us note that using for the leptonic branching fractions the values denoted by PDG as "our fit" [44] we obtain (see Appendix):

$$(R_\tau)^{avg}_{exp} = 3.535 \pm 0.037. \tag{29}$$

which is 1.5σ smaller than (23). Using this value we find in NNLO:

$$\Lambda_{\overline{MS}}^{(3)} = 331\text{MeV(opt)}, \quad \alpha_s^{\overline{MS}}(m_\tau^2) = 0.309, \quad \alpha_s^{\overline{MS}}(m_Z^2) = 0.1162, \tag{30}$$

with the same theoretical and experimental uncertainties as before.

To have a complete picture of theoretical uncertainties we give also the result of a fit of $\Lambda_{\overline{MS}}$ to the experimental value of R_τ, obtained with the improved NLO predictions optimized according to the PMS prescription. Using the NLO expression with $r_1 = -0.76$ we obtain:

$$\Lambda_{\overline{MS}}^{(3)} = 413(\text{opt}) \pm 32(\text{exp})\,\text{MeV}. \tag{31}$$

Using then the two-loop renormalization group equation we find:

$$\alpha_s^{\overline{MS}}(m_\tau^2) = 0.316 \pm 0.013(\text{exp}), \tag{32}$$

$$\alpha_s^{\overline{MS}}(m_Z^2) = 0.1209 \pm 0.0018(\text{exp}). \tag{33}$$

Summarizing, we have evaluated the perturbative QCD corrections to the semileptonic decay width of the tau lepton, using the contour integral representation and evaluating numerically the running coupling constant on the contour. We obtained results in a broad class of renormalization schemes. We discussed the problem of obtaining the optimal predictions in the improved evaluation, and we found predictions for δ_τ in the scheme preferred by the Principle of Minimal Sensitivity, both

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in NLO and in NNLO. Using a specific condition to eliminate the schemes with unnaturally large expansion coefficients we obtained a quantitative estimate of the ambiguities in δ_τ arising from the freedom of choice of the RS. We expect that this estimate would be useful in combining the QCD results from the R_τ with other experimental constraints on QCD. We found that the improved expression for δ_τ is much more stable with respect to change of RS compared to the conventional perturbative NNLO expression. Nevertheless, we found that the RS dependence ambiguities in the fit of $\Lambda^{(3)}_{\overline{MS}}$ to the experimental data for δ_τ are at least of the order of uncertainties arising from the present accuracy of experimental determinations of R_τ.

Note added

Soon after this paper was posted on the bulletin board, two papers appeared [46] [47] in which determination of α_s from the tau decay is discussed in detail. We make two comments about these papers. First, the value of α_s obtained in [46] is almost identical to the value given in our paper. However, this value was obtained using $(R_\tau)_{exp} = 3.56 \pm 0.03$, which is 1σ lower than the value we have used in our analysis. Secondly, the main source of the theoretical uncertainly estimated in [46] [47] appears to be the contribution from the yet uncalculated $O(\alpha_s^4)$ corrections. Such an estimate necessarily involves some speculative assumptions on the magnitude of the higher order terms. Also, the higher order contributions would be RS dependent, with the freedom of choice of the RS characterized by *three* arbitrary parameters in the four-loop order. It is straightforward to extend the method used in our paper to the four-loop case in order to obtain a quantitative estimate of the resulting RS ambiguity.

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Appendix

In this appendix we make some remarks on the experimental determination of R_τ. First, let us note that the experimental value of R_τ may be obtained from *three* independent measurements: from the electronic and muonic branching ratios in the tau lepton decay, and from the total width of the tau lepton. To obtain R_τ from the electronic branching ratio we take:

$$(R_\tau)_{exp}^{B_e} = \frac{1}{(B_e)_{exp}} - 1 - \left(\frac{B_\mu}{B_e}\right)_{th},$$

where according to formulas given in [25], with slightly updated parameters, we should take:

$$\left(\frac{B_\mu}{B_e}\right)_{th} = 0.972568.$$

To obtain R_τ from the muonic branching ratio we take:

$$(R_\tau)^{B_\mu}_{exp} = \frac{1}{(B_\mu)_{exp}}\left(\frac{B_\mu}{B_e}\right)_{th} - 1 - \left(\frac{B_\mu}{B_e}\right)_{th}.$$

Finally, to obtain R_τ from the total decay width we take:

$$(R_\tau)^{\Gamma_\tau}_{exp} = \frac{(\Gamma_{tot})_{exp}}{(\Gamma_e)_{th}} - 1 - \left(\frac{\Gamma_\mu}{\Gamma_e}\right)_{th},$$

with

$$(\Gamma_e)_{th} = (0.40341 \pm 0.00057) \times 10^{-12}\text{GeV},$$

obtained from the formulas given in [25] with updated parameters.

As may be expected, these three determinations do not give exactly the same value. Therefore we take for $R = < R > \pm \delta R$ a weighted average, according to the prescription:

$$< R > = \frac{\sum_i w_i R_i}{\sum_i w_i}, \quad \delta R = \left(\sum_i w_i\right)^{-1/2},$$

where $w_i = (\delta R_i)^{-2}$.

Secondly, one has to be careful which set of the world averaged experimental data one uses in the phenomenological analysis. The Particle Data Group [44] gives in fact *two* sets of values for the leptonic branching fractions. One set is a straightforward world average of the data:

$$(B_e)_{exp} = 0.1790 \pm 0.0017, \quad (B_\mu)_{exp} = 0.1744 \pm 0.0023,$$

This set is called by PDG "our average." Another set,

$$(B_e)_{exp} = 0.1801 \pm 0.0018, \quad (B_\mu)_{exp} = 0.1765 \pm 0.0024,$$

called "our fit," is a result of a global fit to the tau lepton branching fractions, taking into account the relevant constraints. Taking "our average" of PDG for the leptonic branching fractions we obtain:

$$(R_\tau)^{B_e}_{exp} = 3.614 \pm 0.053, \quad (R_\tau)^{B_\mu}_{exp} = 3.604 \pm 0.074.$$

Taking the present world average [44] for the total decay width:

$$(\Gamma_{tot})_{exp} = \frac{\hbar}{(\tau_\tau)_{exp}} = (2.227 \pm 0.023) \times 10^{-12}\text{GeV},$$

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we find:
$$(R_\tau)_{exp}^{\Gamma_\tau} = 3.548 \pm 0.065.$$

(For τ_τ the PDG also gives two values — in this case it seems however clear that one should take the value in which results of older experiments have been corrected for the change in the experimental number for m_τ.) Taking a weighted average of three values for R_τ we find:

$$(R_\tau)_{exp}^{avg} = 3.591 \pm 0.036.$$

This value has been further used in our paper to obtain constraints on $\Lambda_{\overline{MS}}^{(3)}$.

It has to be emphasized however, that if we use "our fit" of PDG for the leptonic branching fractions, we find:

$$(R_\tau)_{exp}^{B_e} = 3.525 \pm 0.055, \quad (R_\tau)_{exp}^{B_\mu} = 3.538 \pm 0.075.$$

Together with the value obtained from the total decay width this gives a weighted average of:

$$(R_\tau)_{exp}^{avg} = 3.535 \pm 0.037.$$

Figure Captions

Fig.1. Perturbative QCD predictions for δ_τ as a function of $m_\tau/\Lambda_{\overline{MS}}^{(3)}$, obtained in NLO and NNLO with the conventional expansion (dashed lines) and with the numerical evaluation of the contour integral (solid lines).

Fig.2. Perturbative prediction for δ_τ, obtained from the numerical evaluation of the contour integral for $\Lambda_{\overline{MS}}^{(3)} = 310$ MeV, as a function of the parameters r_1 and c_2 determining the RS. Note that the actual variable on the vertical axis is $c_2 - c_1 r_1$. The regions of scheme parameters satisfying the condition (5) with $l = 2$ and $l = 3$ are also indicated. At the critical point inside the $l = 2$ region we have $\delta_\tau = 0.1649$.

Fig.3. Same as in Fig.2, but for various values of $\ln(m_\tau/\Lambda_{\overline{MS}}^{(3)})$.

Fig.4. Perturbative prediction for δ_τ, obtained from the numerical evaluation of the contour integral representation, as a function of $m_\tau/\Lambda_{\overline{MS}}^{(3)}$. The thick lines denote the preferred NNLO predictions (i.e. obtained in the scheme with $r_1 = 0$ and $c_2 = 1.5\rho_2$ — see text) and the optimal NLO predictions (obtained with $r_1 = -0.76$). Also indicated are the maximal and minimal values obtained in NNLO when the scheme parameters are varied over the $l = 2$ (dashed lines) and $l = 3$ (dash-dotted lines) allowed regions. For comparison we show the experimental constraint on the value of δ_τ^{pert}.

$\ln(m_\tau/\Lambda_{\overline{MS}}^{(3)})$	$(\delta_\tau)_{NNLO}^{opt}$	$\delta_{l=2}^{min}$	$\delta_{l=2}^{max}$	$\delta_{l=3}^{min}$	$\delta_{l=3}^{max}$	$(\delta_\tau)_{NLO}^{opt}$
1.30	.23634	.2293	.2443	.2075	—	.21816
1.35	.22652	.2194	.2322	.2090	—	.20909
1.40	.21711	.2101	.2223	.2052	—	.20056
1.45	.20816	.2014	.2135	.1969	.2401	.19256
1.50	.19968	.1932	.2050	.1890	.2209	.18505
1.55	.19168	.1856	.1971	.1817	.2055	.17800
1.60	.18414	.1785	.1895	.1749	.1928	.17139
1.65	.17705	.1718	.1823	.1685	.1853	.16518
1.70	.17039	.1656	.1754	.1625	.1783	.15935
1.75	.16413	.1596	.1689	.1568	.1718	.15387
1.80	.15824	.1541	.1628	.1515	.1655	.14871
1.85	.15271	.1489	.1569	.1465	.1596	.14386
1.90	.14751	.1440	.1514	.1418	.1540	.13928

Table 1: Numerical values of the preferred NNLO predictions for δ_τ (i.e. obtained in the scheme with $r_1 = 0$ and $c_2 = 1.5\rho_2$ — see text), and the maximal and minimal values obtained after variation of the scheme parameters in the allowed region corresponding to $l = 2$ and $l = 3$, for several values of $\ln(m_\tau/\Lambda_{\overline{MS}}^{(3)})$. For comparison also the values of the preferred NLO predictions are given ($r_1 = -0.76$).
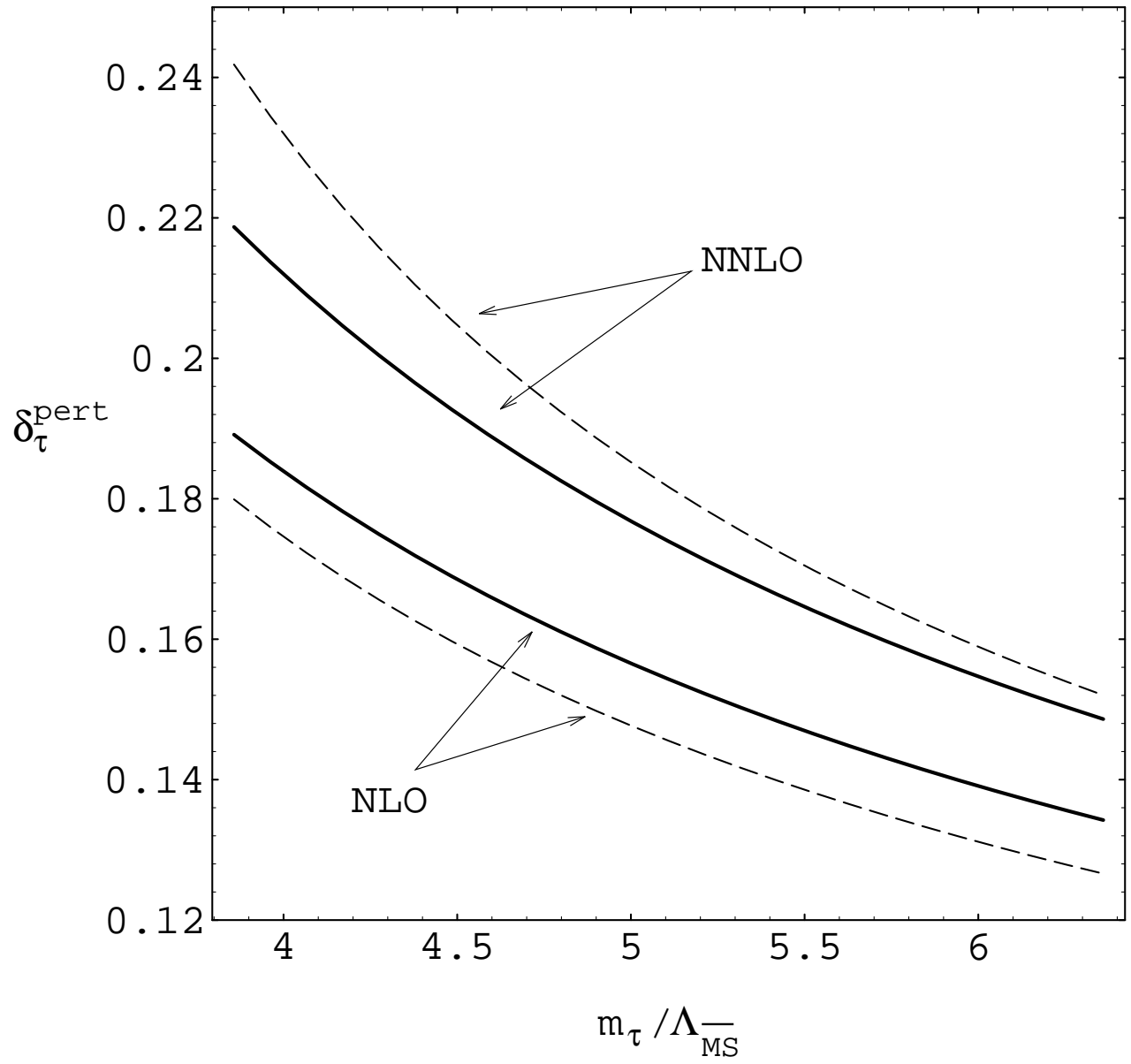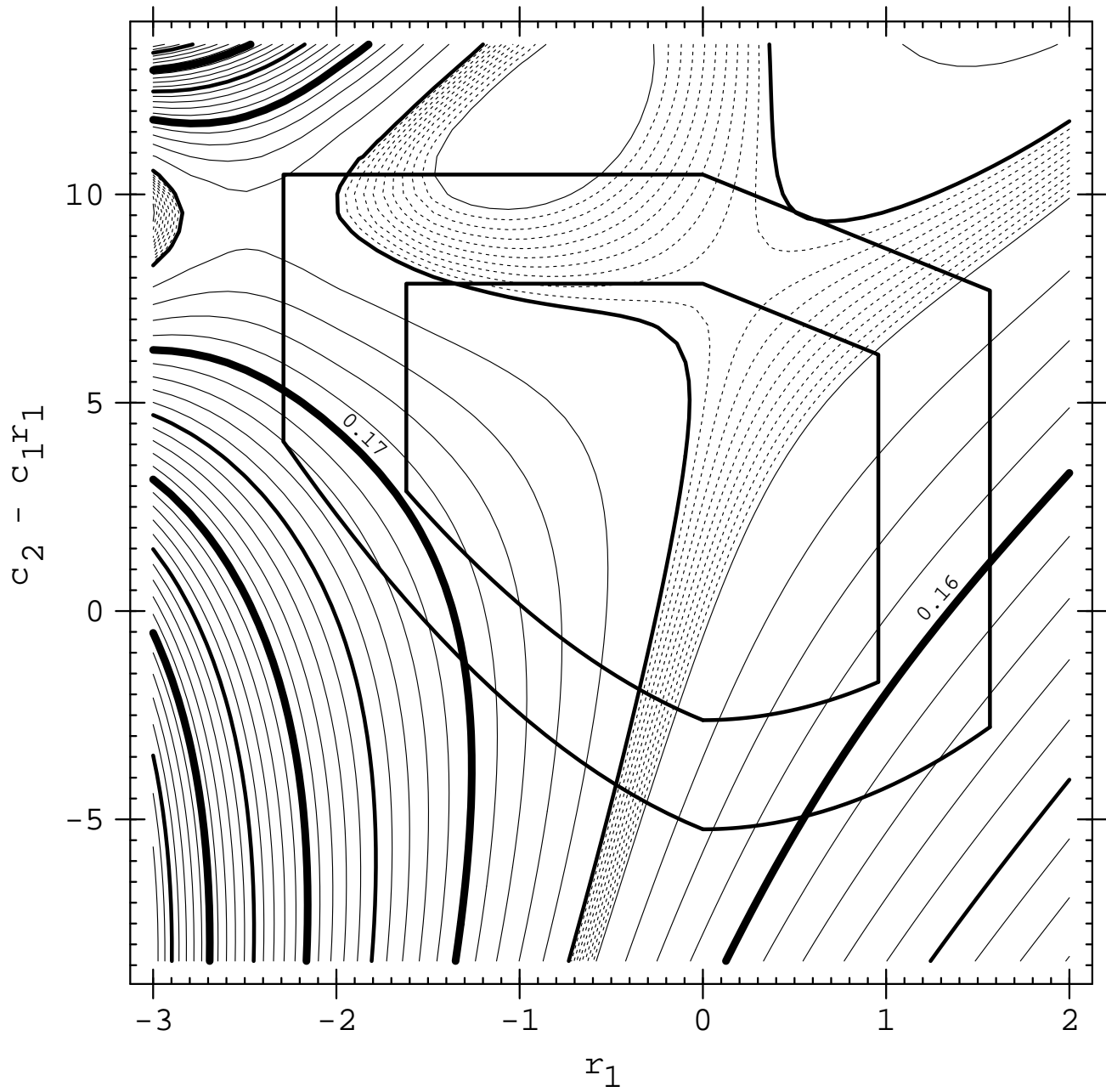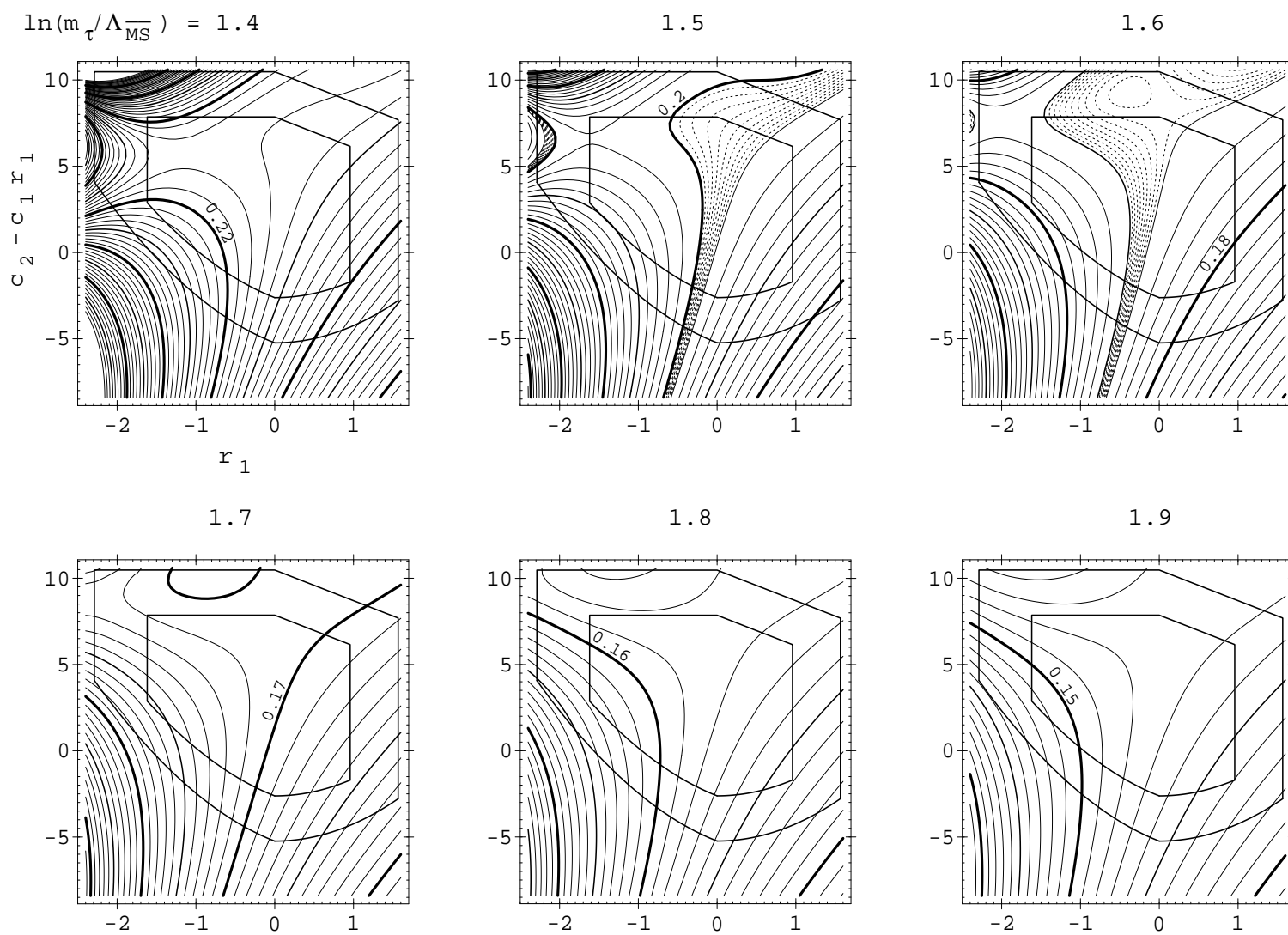
Fig.1

Fig.2

Fig.3

Fig.4